Measurement-theory approach to an internal energy function

1981 J. Phys. A: Math. Gen. 14 303

(http://iopscience.iop.org/0305-4470/14/2/007)

J. Phys. A: Math. Gen. 14 (1981) 303–315. Printed in Great Britain

Measurement-theory approach to an internal energy function

T Nakagomi

Department of Physics, University of Kyoto, Kyoto, Japan

Received 31 August 1979, in final form 21 July 1980

Abstract. It is shown for gaseous systems that the internal energy function can be uniquely constructed solely on the basis of a set of properties of the final temperature reached in the equilibration process due to thermal contact between two gaseous systems with a fixed density. A formulation of a theory of measurement is used, in which an additive quantity, such as mass or length, can be defined as a unique numerical representation of a certain algebraic structure equipped with two operations *addition* (+) and *qualitative comparison* (an order relation $\leqslant$).

1. Introduction

This paper is concerned with such problems as whether the concept of internal energy can be characterised solely by the conservation law in thermal interaction of two thermodynamic systems without any use of the concept of mechanical energy, and if it is possible, what properties of the thermal interaction are essential to the construction of the internal energy function. These problems originate in Giles' axiomatic thermodynamics (Giles 1964, cf Landsberg 1970), in which he derived a thermodynamic structure composed of an entropy function and a set of conserved quantities from an axiom system representing some characteristics of the thermodynamic process. His axioms are, however, quite general and abstract, and it is difficult to designate the internal energy from the set of conserved quantities, whose number is infinite except in a trivial case. His construction of an energy function (Giles 1964 p 105) seems to be physically vague, because he persists in a standpoint of a 'primitive observer' and does not use any parameter to specify the state of a thermodynamic system. This paper does not take so primitive a standpoint. We focus attention on a gaseous system with a fixed density whose states are specified by two parameters, absolute temperature and mole number, and aim at constructing the internal energy function solely on the basis of some properties of the final temperature reached in the equilibration process due to thermal contact between two such gaseous systems.

The essence of the method used here is a formulation of a theory of measurement which can typically be illustrated in the case of the measurement of mass by a balance. The balance is used only to compare the masses of two objects. Let us write $a < b$ if the object b is heavier than the object a, $a \sim b$ if a is balanced with b, $a \leqslant b$ if $a \sim b$ or $a < b$ and $a + b$ when we put a and b together on one side of a balance. Then experience suggests the following: (i) $a \leqslant b \Leftrightarrow a + c \leqslant b + c$, and (ii) for any two objects a and b, we have $b < na$ for sufficiently large positive integers n. Fact (ii) corresponds to Archimedes' axiom of real numbers (Stečkin 1963 p 15), and such a structure of $\leqslant$ and $+$

0305-4470/81/020303+13$01.50

with conditions (i) and (ii) leads to a well known mathematical proposition that there exists uniquely up to a positive multiplicative constant a real function M such that (1) $a \leqslant b \Leftrightarrow M(a) \leqslant M(b)$, (2) $M(a+b) = M(a) + M(b)$, and (3) $M(a) \not\equiv 0$. Thus the numerical scale for mass is uniquely determined from the qualitative comparison if an object is specified as the unit of the scale. The idea of constructing an additive numerical scale on the basis of qualitative relations or an algebraic structure with the operations + and $\leqslant$ was discussed long ago by von Helmholtz (1930) and Weyl (1949, cf Giles 1964 p 3) from a philosophical viewpoint concerning a general process of measurement, and the idea has been developed as a 'theory of measurement' in relation mainly to psychological or economic problems. Giles' thermodynamics is also based on the idea. There are comprehensive reviews by Krantz *et al* (1971) and by Pfanzagl (1971).

In § 3 we list some physically acceptable properties of a function which connects two initial states to the final temperature produced by them through thermal contact, and then construct on the basis of the properties a structure equipped with an addition + and an ordering $\leqslant$ which can be interpreted respectively as the union of two systems through thermal contact and mixing and as a qualitative comparison of internal energy. Finally, the structure is shown to satisfy conditions (i) and (ii) with some modifications, and we necessarily obtain a unique additive numerical scale for internal energy in the same way as in the case of mass. The numerical scale has the following properties: (a) It satisfies a conservation law (the additivity, in this case, means a conservation law), and (b) it decreases to zero as the absolute temperature approaches zero. In § 2 we set up a mathematical theory of the structure of $\leqslant$ and + in a form suitable for application to our problem.

2. Additive ordered structure

Let $\mathscr{S}$ be a non-empty set associated with an operation + and a relation $\leqslant$. We define an additive ordered structure $(\mathscr{S}, +, \leqslant)$ by the following four axioms (in this section $a, b, c, \ldots$ indicate the elements in $\mathscr{S}$):

Axiom A1. $\mathscr{S}$ is an additive semigroup with the operation +, that is (i) $a+b \in \mathscr{S}$, (ii) $a+b = b+a$, and (iii) $(a+b)+c = a+(b+c)$.

Axiom A2. The relation $\leqslant$ is a total quasi-order in $\mathscr{S}$, that is (i) for any $a, b \in \mathscr{S}$ either $a \leqslant b$ or $b \leqslant a$ holds, (ii) $a \leqslant a$, and (iii) $a \leqslant b$ and $b \leqslant c \Rightarrow a \leqslant c$.

Axiom A3. $a \leqslant b \Rightarrow a+c \leqslant b+c$.

Definition 2.1. (i) $a \sim b \Leftrightarrow a \leqslant b$ and $b \leqslant a$. (ii) $a < b \Leftrightarrow a \leqslant b$ and $a \nsim b$, where $a \nsim b$ means that $a \sim b$ is not true. (iii) $\theta \equiv \{a \in \mathscr{S};\ a+c \sim c$ for all $c \in \mathscr{S}\}$. We call the elements in θ null elements of $\mathscr{S}$.

Axiom A4. If $a \notin \theta$ and $b \in \mathscr{S}$, then there exists a positive integer N such that $b < na$ for all integers $n > N$.

Theorem 2.1. If $(\mathscr{S}, +, \leqslant)$ is an additive ordered structure, then there exists a mapping $M: \mathscr{S} \to \mathbb{R}$ (the real line), unique up to a positive multiplicative constant, with the following properties:

B1. $M(a+b)=M(a)+M(b)$ (2.1)

B2. $a \leqslant b \Rightarrow M(a) \leqslant M(b)$ (2.2)

B3. $a \in \theta \Leftrightarrow M(a)=0$ (2.3)

The proof of this theorem is given in the Appendix.

Definition 2.2. We call the function M defined in the above theorem a measure function of the structure $(\mathscr{S}, +, \leqslant)$. The measure function M is faithful iff the converse of B2 (i.e. $M(a) \leqslant M(b) \Rightarrow a \leqslant b$) holds.

Remark. Since the quasi-order $\leqslant$ is total, $a \nleqslant b$ (the negation of $a \leqslant b$) is equivalent to $b < a$. Therefore the converse of B2 is equivalent to

$$a < b \Rightarrow M(a) < M(b), \tag{2.4}$$

which implies that the following condition (i) is necessary and sufficient and (ii) is sufficient for the measure function to be faithful:

(i) Given a, b and c, if $a < b$ then $na + c < nb$ for sufficiently large positive integers n.

(ii) If $a < b$ then there exists $c \notin \theta$ such that $a + c \leqslant b$.

3. Internal energy of gaseous systems

Consider a gaseous system of one component whose density is fixed. Its equilibrium states are specified by the absolute temperature T and the mole number n (or mass or volume; these are proportional to each other since the density is fixed). We define the set of states by

$$\mathscr{S} \equiv \{(T, n);\ T \geqslant 0,\ n > 0\}. \tag{3.1}$$

If two such systems of states (T, n) and (T', n') are put in thermal contact with each other, their temperatures will change and reach an equilibrium temperature $f(T, n; T', n')$. Then by removing the wall between them, we obtain a new equilibrium state $(f(T, n;\ T', n'), n+n')$. The explicit form of the temperature–composition function $f(T, n; T', n')$ depends upon the kind of gas, but we can assume general properties as given below:

D1. $f(T, n; T', n') = f(T', n'; T, n)$

D2. $f(f(T, n; T', n'), n+n'; T'', n'') = f(T, n; f(T', n'; T'', n''), n'+n'')$

D3. $f(T, n; T', n')$ is (i) continuous for each variable of T, n, T', and n', and (ii) strictly monotonic increasing for T(resp. T') with n, T'(resp. T) and n' fixed.

D4. $T < T' \Rightarrow T < f(T, n; T'n') < T'$

D5. $f(T, n; T', n') \to \infty (T \to \infty)$

D6. $f(T, n; T', n') \to T (n \to \infty)$

D7. $f(T, n; T', n') \to T' (n \to 0)$

On the other hand, these properties are considered to be due to the existence of a conserved quantity called internal energy. In the case of an ideal gas, the internal energy U is given by $U = \alpha n T$ (α is a positive constant), whose conservation law provides the function f as†

$$f(T, n; T', n') = (nT + n'T')/(n + n'). \tag{3.2}$$

It is obvious that equation (3.2) satisfies all the properties D1–D7. In general it can be proved that if the internal energy function U has the form $U(n, T) = n\Psi(T)$, then the temperature–composition function f derived from it has all the properties listed above, where Ψ is a strictly monotonic increasing continuous function with $\Psi(0) = 0$.

In this section, we treat the converse problem, namely, whether we can derive the concept of internal energy only from the observed properties D1–D7 of f; in other words, whether we can find out an additive ordered structure whose measure function can be interpreted as the internal energy. This question is answered affirmatively, and the results are summed up by the definitions and theorems below.

In the following we assume that $\mathscr{S}$ is the set defined by equation (3.1) whose elements (T, n) are called states, and f is a function defined on $\mathscr{S} \times \mathscr{S}$ with properties D1–D7.

Definition 3.1. A binary operation + in $\mathscr{S}$ is defined as

$$(T, n) + (T', n') = (f(T, n; T', n'), n + n'). \tag{3.3}$$

It follows from D1 and D2 that the operation + is commutative and associative.

Definition 3.2. A function U on $\mathscr{S}$ is an internal energy function iff it satisfies the conditions

E1. $U((T, n) + (T', n')) = U(T, n) + U(T', n')$

E2. $U(T, n) > U(0, n) = 0 \qquad (T > 0)$.

This definition characterises the internal energy by two properties whose physical meanings are as follows: E1 is the conservation law and E2 represents the fact that the state (T, n) $(T > 0)$ is obtained by heating the state $(0, n)$, and no heat can be drawn from the state $(0, n)$.

Definition 3.3. We define two relations $\approx$ and $\leqslant$ as

(i) $(T, n) \approx (T', n') \Leftrightarrow$ There exist two states $(0, n_1)$ and $(0, n_2)$ such that $(0, n_1) + (T, n) = (0, n_2) + (T', n')$.

(ii) $(T, n) \leqslant (T', n') \Leftrightarrow$ There exists a state (T'', n'') such that $(T'', n'') + (T, n) \approx (T', n')$.

If an internal energy function U exists, we have $(T, n) \approx (T', n') \Rightarrow U(T, n) = U(T', n')$, and $(T, n) \leqslant (T', n') \Rightarrow U(T, n) \leqslant U(T', n')$, hence the relations $\approx$ and $\leqslant$ are considered to make qualitative comparisons with respect to the internal energy.

The relation $\approx$ is an equivalence relation. The reflexivity and symmetricity are obvious. To prove transitivity we notice the following fact implied by D4 and D3(i): For any two states (T, n) and (T, n') with an equal temperature T,

$$f(T, n; T, n') = T \tag{3.4}$$

or equivalently

$$(T, n) + (T, n') = (T, n + n'). \tag{3.5}$$

† According to Bergthorsson (1978), some pioneers in thermodynamics pointed out that a formula of the final temperature such as equation (3.2) indicates a conservation law.

Suppose that $(T, n) \approx (T', n')$ and $(T', n') \approx (T'', n'')$. Then, by definition, there exist four positive numbers n_1, n_2, n_3, n_4 such that

$$(0, n_1) + (T, n) = (0, n_2) + (T', n') \tag{3.6}$$

$$(0, n_3) + (T', n') = (0, n_4) + (T'', n''). \tag{3.7}$$

Add $(0, n_3)$ to equation (3.6), and $(0, n_2)$ to equation (3.7), then, by equation (3.5), we have

$$(0, n_1 + n_3) + (T, n) = (0, n_2 + n_3) + (T', n') = (0, n_4 + n_3) + (T'', n'') \tag{3.8}$$

which implies $(T, n) \approx (T'', n'')$.

Theorem 3.1. With the above definitions of + and $\leqslant$ the system $(\mathscr{S}, +, \leqslant)$ is an additive ordered structure, and its measure function is faithful and satisfies the conditions E1 and E2 in definition 3.2.

Corollary 3.2. An internal energy function U exists uniquely up to a positive multiplicative constant, and has the form

$$U(T, n) = n\Psi(T), \tag{3.9}$$

where $\Psi(T)$ is a strictly increasing continuous function of $T \geqslant 0$ with $\Psi(0) = 0$.

Corollary 3.3. By the use of Ψ, the temperature–composition function f is written in the form

$$f(T, n', T', n') = \Psi^{-1}\left(\frac{n}{n+n'}\Psi(T) + \frac{n'}{n+n'}\Psi(T')\right) \tag{3.10}$$

where Ψ^{-1} is the inverse of Ψ. Conversely, if f has the form (3.10) with a strictly increasing continuous function $\Psi(T)(T \geqslant 0, \Psi(0) = 0)$, then the properties D1–D7 are all fulfilled.

Remark. Corollary 3.3 means that equation (3.10) is the general solution of the functional equations D1 and D2 with supplementary conditions D3–D7. For a general theory of this type of functional equation see Aczél (1966).

Proof of theorem 3.1. Let us show that the system $(\mathscr{S}, +, \leqslant)$ satisfies axioms A1–A4.
A1. This follows from D1 and D2.
A2. Part (ii) of A2, the reflexivity of $\leqslant$, follows from the fact that

$$(0, n') + (T, n) \approx (T, n) \text{ for any states } (T, n) \text{ and } (0, n') \tag{3.11}$$

Part (iii), the transitivity, can be also easily checked by use of the fact that

$$(T, n) \approx (T', n') \Rightarrow (T'', n'') + (T, n) \approx (T'', n'') + (T', n'). \tag{3.12}$$

The proof of (i) is more complicated. From definition 3.3, $(T, n) \leqslant (T', n') \Leftrightarrow \exists$ three states $(0, n_1)$, $(0, n_2)$ and (T_3, n_3) such that

$$(0, n_2) + (T_3, n_2) + (T, n) = (0, n_1) + (T', n') \tag{3.13}$$

$\Leftrightarrow \exists$ two states $(0, n_1)$, (T_2, n_2) such that

$$(T_2, n_2) + (T, n) = (0, n_1) + (T', n') \tag{3.14}$$

where '$\Leftarrow$' of the second equivalence is checked by adding a state $(0, n)$ to both sides of equation (3.14) and by using equation (3.5). Let us express equation (3.14) by use of the function f. We have that $(T, n) \leqslant (T', n') \Leftrightarrow \exists$ three numbers $n_1 > 0$, $n_2 > 0$ and $T_2 \geqslant 0$ such that

$$n_2 + n = n_1 + n', \qquad f(T_2, n_2; T, n) = f(0, n_1; T', n'). \tag{3.15}$$

By use of the above fact, A2(i) is expressed as follows: For any two states (T_1, n_1) and (T_2, n_2) at least one of the two following systems of equations has a solution such that $x \geqslant 0$, $y > 0$, $z > 0$:

$$\text{F1}\begin{cases} \text{(i)} & y + n_2 = z + n_1 \\ \text{(ii)} & f(x, y; T_2, n_2) = f(0, z; T_1, n_1) \end{cases}$$

$$\text{F2}\begin{cases} \text{(i)} & y + n_1 = z + n_2 \\ \text{(ii)} & f(x, y; T_1, n_1) = f(0, z; T_2, n_2). \end{cases}$$

These will be checked for two cases (a) $n_1 \geqslant n_2$ and $T_1 \geqslant T_2$, and (b) $n_1 > n_2$ and $T_1 < T_2$. The other cases are reduced to these by exchanging the subscripts 1 and 2.

Case (a). This is subdivided into the three cases (i) $T_1 = T_2$, $n_1 = n_2$, (ii) $T_1 = T_2$, $n_1 > n_2$, and (iii) $T_1 > T_2$, $n_1 \geqslant n_2$. In case (i), both F1 and F2 hold for $x = 0$ and $y = z > 0$. In case (ii) F1 holds if we put $z = \alpha > 0$, $y = n_2 - n_1 + \alpha$ and $x = f(0, \alpha; T_1, n_1 - n_2)$ with an arbitrary positive number α, as follows:

$$\begin{aligned} f(x, y; T_2, n_2) &= f(f(0, \alpha; T_1, n_1 - n_2), n_2 - n_1 + \alpha; T_1, n_2) \\ &= f(0, \alpha; f(T_1, n_1 - n_2, T_1, n_2), n_1) \ (\because \text{D2}) \\ &= f(0, \alpha; T_1, n_1) \ (\because (3.4)). \end{aligned}$$

In case (iii), since

$$0 < f(0, n; T_1, n_1) < T_1 \qquad \text{for any } n > 0 \ (\because \text{D4})$$

and

$$f(0, n; T_1, n_1) \to T_1 (n \to 0) \ (\because \text{D7}),$$

we have a small number $\alpha > 0$ such that

$$T_2 < f(0, \alpha; T_1, n_1) < T_1. \tag{3.16}$$

On the other hand, it follows from equation (3.4) and D3 and D5 that the value of $f(x, n_2 - n_1 + \alpha; T_2, n_2)$ increases continuously from T_2 to infinity as x does from T_2 to infinity. Therefore for some value $x = \beta (> T_2)$ we have the equality

$$f(\beta, n_2 - n_1 + \alpha; T_2, n_2) = f(0, \alpha; T_1, n_1) \tag{3.17}$$

which implies that F1 holds for $x = \beta$, $y = n_2 - n_1 + \alpha$ and $z = \alpha$.

Case (b). If $\exists$ a positive number α such that

$$T_1 < f(0, n_1 - n_2 + \alpha; T_2, n_2), \tag{3.18}$$

then from the same consideration as in the case (a iii) it follows that F2 holds for $z = \alpha$, $y = n_1 - n_2 + \alpha$ and some value $x = \beta (> T_1)$.

If inequality (3.18) does not hold for any $\alpha > 0$ we have

$$f(0, n_1 - n_2; T_2, n_2) = \lim_{\alpha \to +0} f(0, n_1 - n_2 + \alpha; T_2, n_2) \leqslant T_1, \tag{3.19}$$

where we have used D3(i). Since $f(x, n_1-n_2; T_2, n_2)$ is a continuous function of x and tends to infinity as $x\to\infty$, inequality (3.19) implies that there exists a non-negative number β such that

$$f(\beta, n_1-n_2; T_2, n_2)=T_1. \tag{3.20}$$

Therefore, for any $\alpha>0$ we have

$$\begin{aligned} f(f(0, \alpha; \beta, n_1-n_2), n_1-n_2+\alpha; T_2, n_2) \\ &= f(0, \alpha; f(\beta, n_1-n_2; T_2, n_2), n_1)\ (\because \text{D2}) \\ &= f(0, \alpha; T_1, n_2)\ (\because (3.20)) \end{aligned}$$

which implies that $x=f(\beta, n_1-n_2; 0, \alpha)$, $y=n_1-n_2+\alpha$, $z=\alpha$ make a solution of F1.
A3. This follows from equation (3.14).
Null elements. In order to obtain the set of all null elements of $\mathscr{S}$ we make two preparations. First we show that

$$(T, n)+(T', n')=(T, n)+(T'', n'')\Rightarrow(T', n')=(T'', n''). \tag{3.21}$$

Suppose that the left-hand side of the above implication holds. Then we have

$$f(T, n; T', n')=f(T, n; T'', n''), \tag{3.22}$$

and

$$n+n'=n+n'', \tag{3.23}$$

which leads to $n'=n''$ and

$$f(T, n; T', n')=f(T, n; T'', n'). \tag{3.24}$$

By applying D3(ii) to (3.24), we have $T'=T''$.

Next we show that

$$(T, n)\sim(T', n')\Leftrightarrow(T, n)\approx(T', n'). \tag{3.25}$$

Suppose $(T, n)\sim(T', n')$, which, by definition 2.1(i), means that $(T', n')\preccurlyeq(T, n)$ and $(T, n)\preccurlyeq(T', n')$. Then $\exists$ four states $(0, n_1)$, (T_2, n_2), $(0, n_3)$ and (T_4, n_4) such that

$$(0, n_1)+(T, n)=(T_2, n_2)+(T', n') \tag{3.26}$$

$$(0, n_3)+(T', n')=(T_4, n_4)+(T, n). \tag{3.27}$$

By adding $(0, n_3)$ to (3.26) and (T_2, n_2) to (3.27), we have

$$\begin{aligned} (0, n_3)+(0, n_1)+(T, n) &= (0, n_3)+(T_2, n_2)+(T', n') \\ &= (T_2, n_2)+(T_4, n_4)+(T, n) \end{aligned} \tag{3.28}$$

which, by (3.21), reduces to

$$(0, n_3)+(0, n_1)=(T_2, n_2)+(T_4, n_4). \tag{3.29}$$

Then

$$f(T_2, n_2; T_4, n_4)=0 \tag{3.30}$$

which implies that the right-hand side of D4 does not hold, consequently $T_2=T_4$. Hence by applying (3.4) to (3.30) we have $T_2=T_4=0$. Thus (3.26) implies $(T, n)\approx(T', n')$. The converse is obvious.

A null element (T_0, n_0) is, then, characterised by the condition

$$(T, n)+(T_0, n_0) \approx (T, n) \qquad \forall \text{ states } (T, n). \tag{3.31}$$

By (3.21), condition (3.31) implies $T_0=0$. Conversely if $T_0=0$ then (3.31) is true. Thus a state (T, n) is a null element if and only if $T=0$.

A4. By definition 2.1(ii) and (3.25) we have

$$(T', n') \prec (T, n) \Leftrightarrow (T', n') \preccurlyeq (T, n) \text{ and } (T', n') \not\approx (T, n) \tag{3.32}$$

$\Leftrightarrow \exists$ two states $(T_1, n_1)(T_1>0)$ and $(0, n_0)$ such that

$$(T_1, n_1)+(T', n')=(0, n_0)+(T, n) \tag{3.33}$$

where '$\Leftarrow$' of the second equivalence follows from G7 in the Appendix. Furthermore, from equation (3.5) we obtain for any integer $N>0$

$$\underbrace{(T, n)+\ldots+(T, n)}_{N} \equiv N(T, n)=(T, Nn). \tag{3.34}$$

Thus, in order to check A4, it suffices to show that given two states $(T, n)(T>0)$ and (T', n'), the following system of equations F3 has a solution $x>0$, $y>0$ and $z>0$ for sufficiently large integers $N>0$:

$$\text{F3}\begin{cases} \text{(i)} \ y+n'=z+Nn \\ \text{(ii)} \ f(x, y; T', n')=f(0, z; T, Nn). \end{cases}$$

In the case $T \geqslant T'$, for any integer $N>n'/n$ $\exists$ a positive number $\beta \geqslant T$ such that

$$f(\beta, Nn-n'; T', n')=T, \tag{3.35}$$

because by D3, D5 and (3.4) the value of $f(x, Nn-n'; T', n')$ increases continuously from T' to infinity as x does from T' to infinity. Then for any $\alpha>0$ we have

$$\begin{aligned} &f(f(0, \alpha; \beta, Nn-n'), Nn-n'+\alpha; T', n') \\ &\quad =f(0, \alpha; f(\beta, Nn-n'; T', n'), Nn) \\ &\quad =f(0, \alpha; T, Nn) \end{aligned} \tag{3.36}$$

which implies that F3 holds for $x=f(0, \alpha; \beta, Nn-n')$, $y=Nn-n'+\alpha$ and $z=\alpha$. Moreover, the positivity of $f(0, \alpha; \beta, Nn-n')$ follows from D4 and $\beta>0$.

In the case $T<T'$, take a positive number β_0 such that $T>\beta_0>0$, then it follows from D6 that $\exists$ an integer $N_0>n'/n$ such that

$$f(\beta_0, Nn-n'; T', n')<T \qquad \forall N>N_0. \tag{3.37}$$

Therefore, by the same reason as used for the derivation of (3.35), for each integer $N>N_0$ $\exists$ $\beta>\beta_0$ such that

$$f(\beta, Nn-n'; T', n')=T. \tag{3.38}$$

Consequently, in the same way as in the above case, we have a solution of F3 as $x=f(0, \alpha; \beta, Nn-n')$, $y=Nn-n'+\alpha$ and $x=\alpha$ for any $N>N_0$, where α is an arbitrary positive constant. The positivity of $f(0, \alpha; \beta, Nn-n)$ also follows from D4 and $\beta>0$.

Measure function. Since all the required axioms have been checked above, the system $(\mathscr{S}, +, \preccurlyeq)$ is an additive ordered structure. Theorem 2.1, then, guarantees the existence and the uniqueness (up to a positive multiplicative constant) of the measure function U of $(\mathscr{S}, +, \leqslant)$, namely

$$\text{(i)} \quad U((T, n)+(T', n')) = U(T, n)+U(T', n') \tag{3.39}$$

$$\text{(ii)} \quad (T', n') \leqslant (T, n) \Rightarrow U(T', n') \leqslant U(T, n) \tag{3.40}$$

$$\text{(iii)} \quad T=0 \Leftrightarrow U(T, n)=0. \tag{3.41}$$

From (3.33), (3.39) and (3.41) we have

$$(T', n') < (T, n) \Rightarrow U(T', n') < U(T, n). \tag{3.42}$$

Therefore U is faithful (see Remark in § 2), and the converse of (3.40) is also true. From those properties of U it is obvious that U satisfies the condiitions E1 and E2 in definition 3.2 and is the internal energy function. This concludes the proof.

Proof of corollaries 3.2 and 3.3. First we show the facts

$$\text{(i)} \quad T' < T \Rightarrow U(T, n) < U(T', n) \tag{3.43}$$

$$\text{(ii)} \quad n_1 < n_2 \text{ and } T \neq 0 \Rightarrow U(T, n_1) < U(T, n_2) \tag{3.44}$$

$$\text{(iii)} \quad U(T, n+n') = U(T, n)+U(T, n'). \tag{3.45}$$

In order to prove (3.43), it is sufficient to show that

$$T' < T \Rightarrow (T', n) < (T, n). \tag{3.46}$$

From D3(ii), we have the inequalities for $T' < T$

$$f(0, z; T', y) < f(0, z; T, y) \leqslant f(x, z; T, y). \tag{3.47}$$

Hence, if $T' < T$, then the equation

$$f(0, z; T', y) = f(x, z; T, y) \tag{3.48}$$

does not hold for any $x \geqslant 0$, $y > 0$ and $z > 0$, which gives (3.46). Fact (3.45) follows from (3.39) and (3.5).

By (3.43) and (3.41)

$$0 < T \Rightarrow 0 < U(T, n). \tag{3.49}$$

By putting $n = n_1$ and $n' = n_2 - n_1$ in (3.45) and by using (3.49), we obtain (3.44).

From (3.44) and (3.45), U is an additive increasing function of n for an arbitrary fixed $T > 0$. Hence, for $T > 0$ it can be written in the form

$$U(T, n) = n\Psi(T) \tag{3.50}$$

where Ψ is a function of $T > 0$. Put $\Psi(0) = 0$, then (3.50) holds for $T \geqslant 0$, because $U(0, n) = 0$. Substitution of (3.50) into (3.43) yields the fact that $\Psi(T)$ is a strictly increasing function of $T \geqslant 0$ and that its inverse Ψ^{-1} exists. By substituting (3.50) into E1 of definition (3.2) and by using (3.3), we obtain the following relationship between f and Ψ:

$$(n+n')\Psi(f(T, n; T', n')) = n\Psi(T)+n'\Psi(T') \tag{3.51}$$

which implies (3.10). Moreover, (3.51) shows that the range of Ψ is given by the interval $[0, \sup_T \Psi(T))$. Therefore, Ψ is continuous, since a monotonic increasing function whose range is an interval cannot have any discontinuous point.

The uniqueness of the internal energy function is attributed to that of the measure function, since E1 and E2 in definition (3.2) imply (3.39), (3.40) and (3.41). The latter half of corollary (3.3) is also straightforwardly checked.

4. Concluding remarks

The discussion in this paper is limited to gaseous systems of one component, and it remains to extend it to deal with the thermal contact and mixing of various kinds of materials including chemical reactions. However, the methods used in §§ 2 and 3 are not restricted to thermodynamics, as can be seen from the circumstance that the theory of measurement has been developed mainly in non-physical fields (Krantz *et al* 1971 and Pfanzagl 1971). It is interesting to apply the methods in a developing field of physics such as chaos physics (Ruelle 1978), in which the formalism of the final temperature or that of the additive ordered structure might be useful to construct a measure for chaos (Oono *et al* 1980).

Acknowledgments

The author wishes to express his gratitude for encouragement received from Professor H Hasegawa. He also wishes to express his thanks to Professor P T Landsberg for supplying the bibliography on the theory of measurement and to Dr D S Broomhead for his help in preparaing the first version of this paper.

Appendix: Proof of theorem 2.1

The proof of this theorem has its essential part in common with that of similar statements listed in Krantz (1971). But there is not the same statement in them, so in this Appendix we give the proof in order that the paper be self-contained.

As a preparation for the proof, we present some facts easily derived from the axioms. $a, b, c, \ldots$ and $a', b', \ldots$ are elements in $\mathscr{S}$, and $n, m, n', \ldots$ are positive integers.

G1. For any a and b, one and only one of the relations $a < b$, $a \sim b$, and $b < a$ is true, and $a \leqslant b \Leftrightarrow a < b$ or $a \sim b$.

G2. The relation $\sim$ is an equivalence relation, that is (i) $a \sim a$, (ii) $a \sim b \Rightarrow b \sim a$, and (iii) $a \sim b$ and $b \sim c \Rightarrow a \sim c$.

G3. $a \sim b \Rightarrow a + c \sim b + c$

G4. $a \leqslant b$, $a \sim a'$ and $b \sim b' \Rightarrow a' \leqslant b'$

G5. $a \leqslant b$ and $c \leqslant d \Rightarrow a + c \leqslant b + d$

G6. $a \leqslant b \Rightarrow na \leqslant nb$

G7. $a \notin \theta \Rightarrow b < a + b$

G8. $a \notin \theta \Rightarrow a + b \notin \theta$

G9. $a \notin \theta \Rightarrow na \notin \theta$

G10. $ma \leqslant na$ and $a \notin \theta \Rightarrow m \leqslant n$

Proof. G1–G6 are easy consequences of the definitions and axioms A1, A2 and A3. The others are derived from these and A4 by the method of *reductio ad absurdum.*

G7. Assume $a \notin \theta$ and $a + b \leq b$, which is, by G1, the negation of $b < a + b$. Then, by A4, $\exists$ a positive integer N such that

$$b < na \qquad \forall\, n > N \tag{A.1}$$

and, applying G6 to $a + b \leq b$, we have

$$na + nb \leq nb. \tag{A.2}$$

From (A.1), (A.2) and G5, it follows that

$$(n+1)b \leq nb \qquad \forall\, n > N, \tag{A.3}$$

which contradicts A4 if $b \notin \theta$. This thus implies $b \in \theta$. However, $b \in \theta$ implies $nb = (n-1)b + b \sim (n-1)b \sim \ldots \sim b$, hence

$$na \sim na + b \sim na + nb \leq nb \sim b. \tag{A.4}$$

Thus we have $na \leq b$, which contradicts (A.1), where we have used G2, G3 and G4.

G8. Let us assume $a \notin \theta$ and $a + b \in \theta$, then by G7,

$$a + b < (a + b) + a \sim a. \tag{A.5}$$

On the other hand, by G7 and G1 we obtain

$$a \leq a + b. \tag{A.6}$$

The combination of (A.5) and (A.6) leads to $a \not\sim a$, which contradicts G2(i).

G9. This is a direct consequence of G8.

G10. Suppose $m > n$, $ma \leq na$ and $a \notin \theta$, then by G7 and G9, we have $na < na + (m-n)a = ma$ contradicting $ma \leq na$.

Proof of theorem 2.1. If $\mathscr{S} = \theta$, the theorem is trivial. Henceforth we assume $\mathscr{S} \neq \theta$.
(1) Uniqueness. Let M and M' be two mappings having the properties B1, B2 and B3. Let us fix an element $e \notin \theta$ and take an arbitrary element $a \in \theta$. Since $M(a) > 0$ and $M(e) > 0$ by B3, we can take two sequences of rational numbers $\{p_k\}$ and $\{q_k\}$ such that

$$0 < p_k < M(a)/M(e) < q_k,$$

$$p_k \uparrow M(a)/M(e) \qquad \text{and } q_k \downarrow M(a)/M(e) \qquad (k \to \infty).$$

Putting $p_k = m_k/n_k$ and $q_k = s_k/r_k$, where m_k, n_k, S_k and r_k are positive integers, we have the inequalities

$$m_k r_k M(e) < n_k r_k M(a) < s_k n_k M(e). \tag{A.7}$$

By applying B1 to (A.7),

$$M(m_k r_k e) < M(n_k r_k a) < M(s_k n_k e). \tag{A.8}$$

Then, by the contraposition of B2 we obtain

$$m_k r_k e < n_k r_k a < s_k n_k e. \tag{A.9}$$

By using B2 for M' and (A.9),

$$M'(m_k r_k e) \leq M'(n_k r_k a) \leq M'(s_k n_k e). \tag{A.10}$$

Then by B1 and B3 of M'

$$m_k/n_k \leq M'(a)/M'(e) \leq s_k/r_k, \tag{A.11}$$

which upon allowing $k \to \infty$ leads to

$$M(a)/M(e) \leq M'(a)/M'(e) \leq M(a)/M(e). \tag{A.12}$$

Therefore,

$$M'(a) = (M'(e)/M(e))M(a). \tag{A.13}$$

This relation holds also for $a \in \theta$ because $M'(a) = M(a) = 0$ for $a \in \theta$.
(2) *Existence.* Let us fix an element $e \notin \theta$, then, by A4, for any a and any m there exists n such that $ma \leq ne$. We write $N(m, a)$ as the minimum of such integers n. If $a \notin \theta$ then, by A4, for any large integer $K > 0$ $\exists$ an integer $L > 0$ such that $Ke < ma$ for all $m > L$, which implies $N(m, a) > K$ for $m > L$. Therefore $N(m, a) \to \infty (m \to \infty)$. Consequently, for sufficiently large m, $N(m, a) \geq 2$, and we have

$$(N(m, a) - 1)e < ma \leq N(m, a)e. \tag{A.14}$$

It is noted that $N(m, a) - 1$ is the maximum of integers n such that $ne < ma$, (cf G1).
By applying G6 to (A.14), we obtain for any m'

$$m'(N(m, a) - 1)e \leq m'ma \leq m'N(m, a)e. \tag{A.15}$$

For sufficiently large m' (such that $N(m', a) \geq 2$) we can exchange m and m' in (A.15):

$$m(N(m', a) - 1)e \leq mm'a \leq mN(m', a)e. \tag{A.16}$$

By applying A2(iii) to (A.15) and (A.16), we obtain

$$m'(N(m, a) - 1)e \leq mN(m', a)e \tag{A.17}$$

and

$$m(N(m', a) - 1)e \leq m'N(m, a)e. \tag{A.18}$$

Then by using G10, we have

$$m'(N(m, a) - 1) \leq mN(m', a) \tag{A.19}$$

and

$$m(N(m', a) - 1) \leq m'N(m, a) \tag{A.20}$$

which leads to

$$\left| \frac{N(m, a)}{m} - \frac{N(m', a)}{m'} \right| < \frac{1}{m} + \frac{1}{m'}.$$

Therefore $\{N(m, a)/m\}$ is a Cauchy sequence, and so the limit $N(m, a)/m (m \to \infty)$ exists. If $a \in \theta$ then, by G7 and G4, $ma \sim a < a + e \sim e$, that is $ma < e$. Thus $N(m, a) = 1$. Consequently, the limit also exists and is equal to zero. Now we can define a mapping M as follows:

$$M(a) \equiv \lim_{m \to \infty} N(m, a)/m. \tag{A.21}$$

In the following, we show that M has the required properties in the order B2, B3, B1.

B2. If $a \leq b$, then by G6 we have

$$ma \leq mb \leq N(m, b)e \tag{A.22}$$

for any m. Then, by the definition of $N(m, a)$ we obtain

$$N(m, a) \leqslant N(m, b). \tag{A.23}$$

Consequently $M(a) \leqslant M(b)$.

B3. If $a \notin \theta$ then, by A4, $\exists$ an integer $m_0 > 0$ such that $e < m_0 a$, hence by G6 we have $ke \leqslant m_0 ka$ for any integer $k > 0$. This implies $N(m_0 k, a) \geqslant k$. Therefore

$$N(m_0 k, a)/m_0 k \geqslant m_0^{-1}. \tag{A.24}$$

By allowing $k \to \infty$, we have

$$M(a) \geqslant m_0^{-1} > 0. \tag{A.25}$$

If $a \in \theta$ then, as shown before, $N(m, a) = 1$ for any m. Therefore $M(a) = 0$.

B1. If $a \notin \theta$ and $b \notin \theta$, then for sufficiently large m,

$$(N(m, a) - 1)e < ma \leqslant N(m, a)e, \tag{A.26}$$

$$(N(m, b) - 1)e < mb \leqslant N(m, b)e. \tag{A.27}$$

By applying G5 to (A.26) and (A.27), we obtain

$$(N(m, a) + N(m, b) - 2)e \leqslant m(a + b) \leqslant (N(m, a) + N(m, b))e. \tag{A.28}$$

Then, by the definition of $N(m, a + b)$, we have

$$N(m, a + b) \leqslant N(m, a) + N(m, b) \tag{A.29}$$

and

$$N(m, a) + N(m, b) - 2 \leqslant N(m, a + b). \tag{A.30}$$

By multiplying both sides of (A.29) and (A.30) by m^{-1} and taking the limit $m \to \infty$, we obtain

$$M(a + b) \leqslant M(a) + M(b) \leqslant M(a + b). \tag{A.31}$$

Therefore

$$M(a + b) = M(a) + M(b). \tag{A.32}$$

In the case $a \in \theta$, it follows from B3 that $M(a) = 0$, and from B2 and the relation $a + b \sim b$ that $M(a + b) = M(b)$. Thus (A.32) holds for any a and b.

References

Aczél J 1966 *Lectures on Functional Equations and Their Applications* (New York: Academic)

Bergthorsson B 1978 The Technical University of Denmark, *Preprint*

Giles R 1964 *Mathematical Foundations of Thermodynamics* (Oxford: Pergamon)

von Helmholtz H 1930 *Counting and Measuring* (New York: Van Nostrand)

Krantz D H, Luce R D, Suppes P and Tversky A 1971 *Foundations of Measurement* vol 1 (New York: Academic)

Landsberg P T 1970 *Proc. of the Int. Conf. Thermodynamics* pp 215–27 (London: Butterworths)

Oono Y, Kohda T and Yamazaki H 1980 *J. Phys. Soc. Jpn.* **48** 738

Pfanzagl J 1971 *Theory of Measurement* (Würzburg: Physia-Verlag)

Ruelle D 1978 *Prog. Theor. Phys. Suppl.* **64** 280

Stečkin S B 1963 *Mathematics, its Content, Method and Meaning* Part 5 Ch XV ed. A D Aleksandrov, A N Kolmogoraov and M A Lavent'ev, transl. K Hirsh (Providence: American Mathematical Society) pp 1–35

Weyl H 1949 *Philosophy of Mathematics and Natural Science* (Princeton: Princeton UP)